



**Sixth Term Examination Papers**

**9470**

**MATHEMATICS 2**

Morning

**THURSDAY 26 JUNE 2014**

Time: 3 hours

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Additional Materials: Answer Booklet  
Formulae Booklet

### **INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

### **INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

**Calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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This question paper consists of 9 printed pages and 3 blank pages.

## Section A: Pure Mathematics

- 1** In the triangle  $ABC$ , the base  $AB$  is of length 1 unit and the angles at  $A$  and  $B$  are  $\alpha$  and  $\beta$  respectively, where  $0 < \alpha \leq \beta$ . The points  $P$  and  $Q$  lie on the sides  $AC$  and  $BC$  respectively, with  $AP = PQ = QB = x$ . The line  $PQ$  makes an angle of  $\theta$  with the line through  $P$  parallel to  $AB$ .

- (i) Show that  $x \cos \theta = 1 - x \cos \alpha - x \cos \beta$ , and obtain an expression for  $x \sin \theta$  in terms of  $x$ ,  $\alpha$  and  $\beta$ . Hence show that

$$(1 + 2 \cos(\alpha + \beta))x^2 - 2(\cos \alpha + \cos \beta)x + 1 = 0. \quad (*)$$

Show that (\*) is also satisfied if  $P$  and  $Q$  lie on  $AC$  produced and  $BC$  produced, respectively. [By definition,  $P$  lies on  $AC$  produced if  $P$  lies on the line through  $A$  and  $C$  and the points are in the order  $A, C, P$ .]

- (ii) State the condition on  $\alpha$  and  $\beta$  for (\*) to be linear in  $x$ . If this condition does not hold (but the condition  $0 < \alpha \leq \beta$  still holds), show that (\*) has distinct real roots.
- (iii) Find the possible values of  $x$  in the two cases (a)  $\alpha = \beta = 45^\circ$  and (b)  $\alpha = 30^\circ$ ,  $\beta = 90^\circ$ , and illustrate each case with a sketch.

- 2** This question concerns the inequality

$$\int_0^\pi (f(x))^2 dx \leq \int_0^\pi (f'(x))^2 dx. \quad (*)$$

- (i) Show that (\*) is satisfied in the case  $f(x) = \sin nx$ , where  $n$  is a positive integer.

Show by means of counterexamples that (\*) is not necessarily satisfied if either  $f(0) \neq 0$  or  $f(\pi) \neq 0$ .

- (ii) You may now assume that (\*) is satisfied for any (differentiable) function  $f$  for which  $f(0) = f(\pi) = 0$ .

By setting  $f(x) = ax^2 + bx + c$ , where  $a$ ,  $b$  and  $c$  are suitably chosen, show that  $\pi^2 \leq 10$ .

By setting  $f(x) = p \sin \frac{1}{2}x + q \cos \frac{1}{2}x + r$ , where  $p$ ,  $q$  and  $r$  are suitably chosen, obtain another inequality for  $\pi$ .

Which of these inequalities leads to a better estimate for  $\pi^2$ ?

- 3** (i) Show, geometrically or otherwise, that the shortest distance between the origin and the line  $y = mx + c$ , where  $c \geq 0$ , is  $c(m^2 + 1)^{-\frac{1}{2}}$ .
- (ii) The curve  $C$  lies in the  $x$ - $y$  plane. Let the line  $L$  be tangent to  $C$  at a point  $P$  on  $C$ , and let  $a$  be the shortest distance between the origin and  $L$ . The curve  $C$  has the property that the distance  $a$  is the same for all points  $P$  on  $C$ .

Let  $P$  be the point on  $C$  with coordinates  $(x, y(x))$ . Given that the tangent to  $C$  at  $P$  is not vertical, show that

$$(y - xy')^2 = a^2(1 + (y')^2). \quad (*)$$

By first differentiating  $(*)$  with respect to  $x$ , show that either  $y = mx \pm a(1 + m^2)^{\frac{1}{2}}$  for some  $m$  or  $x^2 + y^2 = a^2$ .

- (iii) Now suppose that  $C$  (as defined above) is a continuous curve for  $-\infty < x < \infty$ , consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

- 4** (i) By using the substitution  $u = 1/x$ , show that for  $b > 0$

$$\int_{1/b}^b \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)} dx = 0.$$

- (ii) By using the substitution  $u = 1/x$ , show that for  $b > 0$ ,

$$\int_{1/b}^b \frac{\arctan x}{x} dx = \frac{\pi \ln b}{2}.$$

- (iii) By using the result  $\int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{\pi}{2a}$  (where  $a > 0$ ), and a substitution of the form  $u = k/x$ , for suitable  $k$ , show that

$$\int_0^\infty \frac{1}{(a^2 + x^2)^2} dx = \frac{\pi}{4a^3} \quad (a > 0).$$

5 Given that  $y = xu$ , where  $u$  is a function of  $x$ , write down an expression for  $\frac{dy}{dx}$ .

(i) Use the substitution  $y = xu$  to solve

$$\frac{dy}{dx} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point  $(1, 1)$ .

Give your answer in the form of a quadratic in  $x$  and  $y$ .

(ii) Using the substitutions  $x = X + a$  and  $y = Y + b$  for appropriate values of  $a$  and  $b$ , or otherwise, solve

$$\frac{dy}{dx} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point  $(1, 1)$ .

6 By simplifying  $\sin(r + \frac{1}{2})x - \sin(r - \frac{1}{2})x$  or otherwise show that, for  $\sin \frac{1}{2}x \neq 0$ ,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}.$$

The functions  $S_n$ , for  $n = 1, 2, \dots$ , are defined by

$$S_n(x) = \sum_{r=1}^n \frac{1}{r} \sin rx \quad (0 \leq x \leq \pi).$$

(i) Find the stationary points of  $S_2(x)$  for  $0 \leq x \leq \pi$ , and sketch this function.

(ii) Show that if  $S_n(x)$  has a stationary point at  $x = x_0$ , where  $0 < x_0 < \pi$ , then

$$\sin nx_0 = (1 - \cos nx_0) \tan \frac{1}{2}x_0$$

and hence that  $S_n(x_0) \geq S_{n-1}(x_0)$ . Deduce that if  $S_{n-1}(x) > 0$  for all  $x$  in the interval  $0 < x < \pi$ , then  $S_n(x) > 0$  for all  $x$  in this interval.

(iii) Prove that  $S_n(x) \geq 0$  for  $n \geq 1$  and  $0 \leq x \leq \pi$ .

- 7 (i) The function  $f$  is defined by  $f(x) = |x - a| + |x - b|$ , where  $a < b$ . Sketch the graph of  $f(x)$ , giving the gradient in each of the regions  $x < a$ ,  $a < x < b$  and  $x > b$ . Sketch on the same diagram the graph of  $g(x)$ , where  $g(x) = |2x - a - b|$ .

What shape is the quadrilateral with vertices  $(a, 0)$ ,  $(b, 0)$ ,  $(b, f(b))$  and  $(a, f(a))$ ?

- (ii) Show graphically that the equation

$$|x - a| + |x - b| = |x - c|,$$

where  $a < b$ , has 0, 1 or 2 solutions, stating the relationship of  $c$  to  $a$  and  $b$  in each case.

- (iii) For the equation

$$|x - a| + |x - b| = |x - c| + |x - d|,$$

where  $a < b$ ,  $c < d$  and  $d - c < b - a$ , determine the number of solutions in the various cases that arise, stating the relationship between  $a$ ,  $b$ ,  $c$  and  $d$  in each case.

- 8 For positive integers  $n$ ,  $a$  and  $b$ , the integer  $c_r$  ( $0 \leq r \leq n$ ) is defined to be the coefficient of  $x^r$  in the expansion in powers of  $x$  of  $(a + bx)^n$ . Write down an expression for  $c_r$  in terms of  $r$ ,  $n$ ,  $a$  and  $b$ .

For given  $n$ ,  $a$  and  $b$ , let  $m$  denote a value of  $r$  for which  $c_r$  is greatest (that is,  $c_m \geq c_r$  for  $0 \leq r \leq n$ ).

Show that

$$\frac{b(n+1)}{a+b} - 1 \leq m \leq \frac{b(n+1)}{a+b}.$$

Deduce that  $m$  is either a unique integer or one of two consecutive integers.

Let  $G(n, a, b)$  denote the unique value of  $m$  (if there is one) or the larger of the two possible values of  $m$ .

- (i) Evaluate  $G(9, 1, 3)$  and  $G(9, 2, 3)$ .
- (ii) For any positive integer  $k$ , find  $G(2k, a, a)$  and  $G(2k - 1, a, a)$  in terms of  $k$ .
- (iii) For fixed  $n$  and  $b$ , determine a value of  $a$  for which  $G(n, a, b)$  is greatest.
- (iv) For fixed  $n$ , find the greatest possible value of  $G(n, 1, b)$ . For which values of  $b$  is this greatest value achieved?

## Section B: Mechanics

- 9** A uniform rectangular lamina  $ABCD$  rests in equilibrium in a vertical plane with the corner  $A$  in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side  $AB$  at a distance  $d$  from  $A$ . The other end of the string is attached to a point on the wall above  $A$  where it makes an acute angle  $\theta$  with the downwards vertical. The side  $AB$  makes an acute angle  $\phi$  with the upwards vertical at  $A$ . The sides  $BC$  and  $AB$  have lengths  $2a$  and  $2b$  respectively. The coefficient of friction between the lamina and the wall is  $\mu$ .

- (i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d \sin(\theta + \phi) = (\cos \theta + \mu \sin \theta)(a \cos \phi + b \sin \phi). \quad (*)$$

- (ii) How should (\*) be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

- (iii) Find a condition on  $d$ , in terms of  $a$ ,  $b$ ,  $\tan \theta$  and  $\tan \phi$ , which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if  $b(2 \tan \theta + \tan \phi) < a$ .

- 10** A particle is projected from a point  $O$  on horizontal ground with initial speed  $u$  and at an angle of  $\theta$  above the ground. The motion takes place in the  $x$ - $y$  plane, where the  $x$ -axis is horizontal, the  $y$ -axis is vertical and the origin is  $O$ . Obtain the Cartesian equation of the particle's trajectory in terms of  $u$ ,  $g$  and  $\lambda$ , where  $\lambda = \tan \theta$ .

Now consider the trajectories for different values of  $\theta$  with  $u$  fixed. Show that for a given value of  $x$ , the coordinate  $y$  can take all values up to a maximum value,  $Y$ , which you should determine as a function of  $x$ ,  $u$  and  $g$ .

Sketch a graph of  $Y$  against  $x$  and indicate on your graph the set of points that can be reached by a particle projected from  $O$  with speed  $u$ .

Hence find the furthest distance from  $O$  that can be achieved by such a projectile.

- 11** A small smooth ring  $R$  of mass  $m$  is free to slide on a fixed smooth horizontal rail. A light inextensible string of length  $L$  is attached to one end,  $O$ , of the rail. The string passes through the ring, and a particle  $P$  of mass  $km$  (where  $k > 0$ ) is attached to its other end; this part of the string hangs at an acute angle  $\alpha$  to the vertical and it is given that  $\alpha$  is constant in the motion.

Let  $x$  be the distance between  $O$  and the ring. Taking the  $y$ -axis to be vertically upwards, write down the Cartesian coordinates of  $P$  relative to  $O$  in terms of  $x$ ,  $L$  and  $\alpha$ .

- (i) By considering the vertical component of the equation of motion of  $P$ , show that

$$km\ddot{x} \cos \alpha = T \cos \alpha - kmg,$$

where  $T$  is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of  $P$  and  $R$ .

- (ii) Show that  $\frac{\sin \alpha}{(1 - \sin \alpha)^2} = k$ , and deduce, by means of a sketch or otherwise, that motion with  $\alpha$  constant is possible for all values of  $k$ .

- (iii) Show that  $\ddot{x} = -g \tan \alpha$ .

## Section C: Probability and Statistics

- 12** The lifetime of a fly (measured in hours) is given by the continuous random variable  $T$  with probability density function  $f(t)$  and cumulative distribution function  $F(t)$ . The *hazard function*,  $h(t)$ , is defined, for  $F(t) < 1$ , by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

- (i) Given that the fly lives to at least time  $t$ , show that the probability of its dying within the following  $\delta t$  is approximately  $h(t) \delta t$  for small values of  $\delta t$ .
- (ii) Find the hazard function in the case  $F(t) = t/a$  for  $0 < t < a$ . Sketch  $f(t)$  and  $h(t)$  in this case.
- (iii) The random variable  $T$  is distributed on the interval  $t > a$ , where  $a > 0$ , and its hazard function is  $t^{-1}$ . Determine the probability density function for  $T$ .
- (iv) Show that  $h(t)$  is constant for  $t > b$  and zero otherwise if and only if  $f(t) = ke^{-k(t-b)}$  for  $t > b$ , where  $k$  is a positive constant.
- (v) The random variable  $T$  is distributed on the interval  $t > 0$  and its hazard function is given by

$$h(t) = \left( \frac{\lambda}{\theta^\lambda} \right) t^{\lambda-1},$$

where  $\lambda$  and  $\theta$  are positive constants. Find the probability density function for  $T$ .



- 13** A random number generator prints out a sequence of integers  $I_1, I_2, I_3, \dots$ . Each integer is independently equally likely to be any one of  $1, 2, \dots, n$ , where  $n$  is fixed. The random variable  $X$  takes the value  $r$ , where  $I_r$  is the first integer which is a repeat of some earlier integer.

Write down an expression for  $P(X = 4)$ .

- (i) Find an expression for  $P(X = r)$ , where  $2 \leq r \leq n + 1$ . Hence show that, for any positive integer  $n$ ,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right) \frac{2}{n} + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \frac{3}{n} + \dots = 1.$$

- (ii) Write down an expression for  $E(X)$ . (You do not need to simplify it.)

- (iii) Write down an expression for  $P(X \geq k)$ .

- (iv) Show that, for any discrete random variable  $Y$  taking the values  $1, 2, \dots, N$ ,

$$E(Y) = \sum_{k=1}^N P(Y \geq k).$$

Hence show that, for any positive integer  $n$ ,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3^2}{n}\right) + \dots = 0.$$

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